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On the Uniform Convergence of Sine Series with Quasimonotone Coefficients

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In this note, the well known theorem of Chaundy and Jolliffe giving necessary and sufficient conditions for the uniform convergence of a sine series with decreasing coefficients is extended to the case where the coefficients are quasimonotone. A related theorem of Hardy is similarly extended. These results are, in a certain sense, the best possible.

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A sequence (b_n) of positive numbers is said to be quasimonotone if for some $\alpha \geqslant 0$, the sequence (b_n/n^α) is non-increasing. The object of this note is to prove the following two results on sine series with quasimonotone coefficients:

THEOREM 1. *If (b_n) is positive and quasimonotone, then a necessary and sufficient condition either for the uniform convergence of $\sum b_n \sin nx$, or for the continuity of its sum function $f(x)$, is that $nb_n \to 0$.*

THEOREM 2. *If (b_n) is positive and quasimonotone, and $A > 0$, then a necessary and sufficient condition for $f(x) \to \frac{1}{2}\pi A$ as $x \to 0+$, is $nb_n \to A$.*

Theorem 1 extends a result of Chaundy and Jolliffe [1] and Jolliffe [5]. By a theorem of Clunie [2], Theorem 1 is a best possible result in the sense that if (b_n) is not quasimonotone, there are uniformly convergent series $\sum b_n \sin nx$ with $b_n > 0$, for which $nb_n \not\to 0$. A result analogous to Theorem 1 holds for $\sum b_n \sin nx$ uniformly bounded with the conclusion $nb_n = O(1)$. Theorem 2 extends a result of Hardy [3]; see also [4]. This theorem also is the best possible in the same sense as before, as may be seen by considering the series $\sum (b_n + A/n) \sin nx$.

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In the following proofs, K denotes a positive constant not necessarily the same at each occurrence.

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Proof of Theorem 1. *Sufficiency.* Because of the periodicity and oddness of all the terms of the series, it is sufficient to consider the interval $[0, \pi]$. Now at $x=0$ and $x=\pi$, all the terms vanish, so it is enough to take $0<x<\pi$. Since $nb_n \to 0$, then given $\varepsilon>0$, we can find n_0 such that for all $n \geqslant n_0$, $nb_n<\varepsilon$ and $\sum_{k=1}^n kb_k<\varepsilon n$. With $0<x<\pi$, choose $N=[1/x]$, the greatest integer not exceeding $1/x$. Then for $n \geqslant n_0$,

$$\sum_{k=n}^{\infty} b_k \sin kx = \left\{ \sum_{k=n}^{N-1} + \sum_{k=N}^{\infty} \right\} b_k \sin kx = \sum\nolimits_1 + \sum\nolimits_2$$

say, where if $n \geqslant N$, we take $\sum_1 = 0$ and the lower limit of summation in $\sum_2$ as n. If $n<N$, then $n \leqslant N-1$ and

$$|\textstyle\sum_1| \leqslant \displaystyle\sum_{k=n}^{N-1} kb_k x \leqslant x \sum_{k=1}^{N-1} kb_k \leqslant x\varepsilon(N-1)<\varepsilon.$$

Also, by Abel's transformation,

$$\begin{aligned}
|\textstyle\sum_2| &= \left| \displaystyle\sum_{k=N}^{\infty} b_k \sin kx \right| = \left| \sum_{k=N}^{\infty} (b_k/k^\alpha)\cdot k^\alpha \sin kx \right| \\
&= \left| \sum_{k=N}^{\infty} (b_k/k^\alpha - b_{k+1}/(k+1)^\alpha) \sum_{v=1}^{k} v^\alpha \sin vx - b_N/N^\alpha \sum_{v=1}^{N-1} v^\alpha \sin vx \right| \\
&= \sum_{k=N}^{\infty} (b_k/k^\alpha - b_{k+1}/(k+1)^\alpha)\, O(k^\alpha/x) + b_N/N^\alpha\, O(N^\alpha/x),
\end{aligned}$$

since $|\sum_{k=1}^n k^\alpha \sin kx| = O(n^\alpha/x)$. For, again by Abel's transformation,

$$\begin{aligned}
\left| \sum_{k=1}^{n} k^\alpha \sin kx \right| &= \left| \sum_{k=1}^{n} (k^\alpha - (k+1)^\alpha) \sum_{v=1}^{k} \sin vx + (n+1)^\alpha \sum_{v=1}^{n} \sin vx \right| \\
&= \sum_{k=1}^{n} O(k^{\alpha-1})\, O(1/x) + O(n^\alpha/x) \\
&= O(n^\alpha/x), \qquad \alpha>0.
\end{aligned}$$

Hence

$$\left|\sum_{k=N}^{\infty} b_k \sin kx\right| \leqslant KN \sum_{k=N}^{\infty} (b_k/k^\alpha - b_{k+1}/(k+1)^\alpha)\cdot k^\alpha + KNb_N.$$

Now

$$\begin{aligned}\sum_{k=N}^{\infty} (b_k/k^\alpha - b_{k+1}/(k+1)^\alpha)\cdot k^\alpha &= b_N + \sum_{k=N}^{\infty} \{(k+1)^\alpha - k^\alpha\}\, b_{k+1}/(k+1)^\alpha \\ &\leqslant b_N + K \sum_{k=N}^{\infty} b_{k+1}/k+1 \\ &\leqslant b_N + K\varepsilon \sum_{k=N}^{\infty} 1/(k+1)^2 \leqslant b_N + K\varepsilon/N.\end{aligned}$$

Hence $|\sum_{k=N}^{\infty} b_k \sin kx| \leqslant KNb_N + K\varepsilon + KNb_N < 3K\varepsilon$, since $nb_n \to 0$.

Necessity. If $\sum b_n \sin nx$ is uniformly convergent, then for all sufficiently large N, $|\sum_{k=N}^{\infty} b_k \sin kx| < \varepsilon$ and hence $|\sum_{k=N}^{2N} b_k \sin kx| < \varepsilon$. Choosing $x = \pi/4N$, then $\sin k\pi/4N \geqslant 1/\sqrt{2}$ for $k = N, (N+1), \ldots, 2N$. Thus $\sum_{k=N}^{2N} b_k < \varepsilon\sqrt{2}$. Now

$$\begin{aligned}\sum_{k=N}^{2N} b_k = \sum_{k=N}^{2N} k^\alpha b_k/k^\alpha &\geqslant \{b_{2N}/(2N)^\alpha\} \sum_{k=N}^{2N} k^\alpha \\ &\geqslant Kb_{2N}\{(2N)^{\alpha+1} - N^{\alpha+1}\}/(2N)^\alpha \\ &= KNb_{2N}(2^{\alpha+1} - 1)/2^\alpha = KNb_{2N}.\end{aligned}$$

Thus $Nb_{2N} < \varepsilon\sqrt{2}/K$, implying that $nb_n \to 0$.

Remark. If $f(x)$ is continuous, then it is everywhere bounded, and by our earlier remarks, $nb_n = O(1)$. Hence the coefficients b_n, are Fourier coefficients, and $\sum b_n \sin nx$ is uniformly convergent. Thus $nb_n \to 0$.

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Proof of Theorem 2. Since the proof is very similar to that of Theorem 2 of Hardy and Rogosinski [4], we confine ourselves to a few points where the arguments differ slightly. We use the notation of [4], except that θ there is replaced by x. Thus for sufficiency, with $A = 1$ and $c_n = b_n - 1/n$, then

$$\begin{aligned}
&\left|\sum_{k=N+1}^{\infty} c_k \sin kx\right| \\
&\quad=\left|\sum_{k=N+1}^{\infty} (c_k/k^\alpha - c_{k+1}/(k+1)^\alpha)\right. \\
&\quad\quad\left.\times \sum_{v=1}^{k} v^\alpha \sin vx - c_N/N^\alpha \sum_{v=1}^{N} v^\alpha \sin vx\right| \\
&\quad\leqslant\left|\sum_{k=N+1}^{\infty} (b_k/k^\alpha - b_{k+1}/(k+1)^\alpha)\, O(k^\alpha/x)\right. \\
&\quad\quad\left.+\sum_{k=N+1}^{\infty} (1/(k+1)^{\alpha+1} - 1/k^{\alpha+1}) \sum_{v=1}^{k} v^\alpha \sin vx\right| + |c_N/N^\alpha|\, O(N^\alpha/x) \\
&\quad\leqslant KN\varepsilon \sum_{k=N+1}^{\infty} (b_k/k^\alpha - b_{k+1}/(k+1)^\alpha)\, k^\alpha \\
&\quad\quad+\sum_{k=N+1}^{\infty} O(1/k^{\alpha+2})\, O(k^\alpha/x) + O(N\varepsilon|c_N|),
\end{aligned}$$

where we choose $1/\varepsilon N < x < 1/(N-1)\,\varepsilon$. Hence

$$\begin{aligned}
\left|\sum_{k=N+1}^{\infty} c_k \sin kx\right| &\leqslant KNb_{N+1}\varepsilon + KN\varepsilon \sum_{k=N+1}^{\infty} b_{k+1}/k+1 \\
&\quad+ KN\varepsilon \sum_{k=N+1}^{\infty} 1/k^2 + KN\varepsilon\, |(Nb_N - 1)/N| \leqslant 4K\varepsilon.
\end{aligned}$$

For the necessity, since $\sum b_n \sin nx$ is the Fourier series of $f(x)$, then $\sum b_n/n$ converges, and by the proof of Theorem 1, it is clear that $\sum b_n \sin nx$ converges uniformly for $0 < \delta \leqslant x \leqslant \pi - \delta$. Now, following the proof in [4], we have, since (b_n/n^α) decreases,

$$(\sigma-\rho)\, b_\sigma \rho^{1+\alpha}/\sigma^\alpha \leqslant \sum_{\rho+1}^{\sigma} n^{1+\alpha}(b_n/n^\alpha) = \sum_{\rho+1}^{\sigma} nb_n,$$

so that

$$(\sigma-\rho)(\rho/\sigma)^{1+\alpha}\, \sigma b_\sigma \leqslant \sigma - \rho + \pi\varepsilon^2/2x,$$

or

$$\sigma b_\sigma \leqslant (\sigma/\rho)^{1+\alpha}\,(1 + \pi\varepsilon^2/2(\sigma-\rho)\,x).$$

Hence

$$\limsup \sigma b_\sigma \leqslant (1 + (\mu - \lambda)/\lambda)^{1+\alpha} (1 + \pi\varepsilon^2/2(\mu - \lambda))$$
$$< (1 + 4\varepsilon/\pi)^{1+\alpha} (1 + \pi\varepsilon),$$

so $\limsup \sigma b_\sigma \leqslant 1$, and a similar argument shows that $\liminf \sigma b_\sigma \geqslant 1$.

Remark. That the series is a Fourier series, follows from the remark above. For $f(x)$ is continuous for $x \in (0, \delta)$, where $\delta > 0$, and hence is bounded for all $x \in (-\delta, \delta)$.

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